

Thermodynamic instability of nonlinearly charged black holes in gravity's rainbow

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Motivated by the violation of Lorentz invariance in quantum gravity, we study black hole solutions in gravity's rainbow in context of Einstein gravity coupled with various models of nonlinear electrodynamics. We regard an energy dependent spacetime and obtain related metric functions and electric fields. We show that there is an essential singularity at the origin which is covered with an event horizon. We also compute the conserved and thermodynamical quantities and examine the validity of the first law of thermodynamics in the presence of rainbow functions. Finally, we investigate thermal stability conditions for these black hole solutions in context of canonical ensemble. We show that thermodynamical structure of the solutions depends on the choices of nonlinearity parameters, charge and energy functions.

I. INTRODUCTION

One of the interesting dreams of physicists is finding a consistent quantum theory of gravity. Although there are a lot of attempts to join gravity and quantum theories together, there is no complete description of the quantum gravity. On the other hand, it has been shown that the violation of Lorentz invariance is an essential primitive rule to construct a quantum theory of gravity. The Lorentz invariance violation may be expressed in form of modified dispersion relations [1–5]. Indeed, regarding various approaches of quantum gravity, there are evidences which show that the Lorentz symmetry might be violated in the ultraviolet limit [6–10], and then it only holds in the infrared limit of quantum theory of gravity. Since the standard energy-momentum dispersion relation enjoys the Lorentz symmetry, it is expected to modify this relation in the ultraviolet limit. In fact, it has been observed that such modification to the standard energy-momentum relation occurs in some models based on string theory [1], spin-network in loop quantum gravity (LQG) [2], spacetime foam [4], the discrete spacetime [5], Horava-Lifshitz gravity [11, 12], ghost condensation [13], non-commutative geometry [3, 14], and double special relativity.

In doubly special relativity, there are two fundamental constants; the velocity of light and the Planck energy. In this theory, it is not possible for a particle to attain energy and velocity larger than the Planck energy and the velocity of light, respectively. The doubly special relativity has been generalized to curved spacetime, and this doubly general theory of relativity is called gravity's rainbow [15, 16]. In gravity's rainbow, the energy of the test particle affects the geometry of spacetime. It means that gravity has different effects on the particles with various energies. Hence, the geometry of spacetime is represented by a family of energy dependent metrics forming a rainbow of metrics. The gravity's rainbow can be constructed by considering following deformation of the standard energy-momentum relation

$$E^2 f^2(\varepsilon) - p^2 g^2(\varepsilon) = m^2, \quad (1)$$

where $\varepsilon = E/E_P$ and E_P is the Planck energy. The functions $f(\varepsilon)$ and $g(\varepsilon)$ are called rainbow functions and they are phenomenologically motivated. The rainbow functions are required to satisfy

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 1, \quad \lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 1. \quad (2)$$

where this condition ensures that we have the standard energy-momentum relation in the infrared limit. It is worthwhile to mention that the spacetime is probed at an energy E , and by definition this can not be greater than the Planck energy E_P . It means that if a test particle is used to probe the geometry of spacetime, then E is the energy of that test particle, and so E can not become larger than E_P [17]. It is worth mentioning that such justification is based on the Standard model of particle physics. In other words, if a particle is described by standard model, the upper limit of the Planck energy is enforced and energy functions will have to satisfy mentioned condition. Whereas, in trans-planckian physics, such condition could be violated. It means that the particle probing spacetime could acquire

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energies larger than the Planck energy. Such property has been considered and employed in number of papers [18–23]. This consideration requires modifications in structure of the energy functions as well. However, we will conduct our study with consideration of the standard model and mentioned conditions for energies that particle could acquire. Now, it is possible to define an energy dependent deformation of the metric $\hat{\mathbf{g}}$ as [17]

$$\hat{\mathbf{g}} = \eta^{\mu\nu} e_\mu(E) \otimes e_\nu(E), \quad (3)$$

where

$$e_0(E) = \frac{1}{f(\varepsilon)} \hat{e}_0, \quad e_i(E) = \frac{1}{g(\varepsilon)} \hat{e}_i, \quad (4)$$

in which the hatted quantities refer to the energy independent frame.

In recent years, the effects of gravity's rainbow have been investigated in the context of black hole thermodynamics in literature [24–27]. The modification in the thermodynamics of black rings and other black objects in the context of gravity's rainbow has been investigated in [28, 29]. In addition, the hydrostatic equilibrium equation in Einstein gravity's rainbow has been obtained and the maximum mass of neutron stars has been investigated in Ref. [30]. As we consider black holes in gravity's rainbow, the energy E corresponds to the energy of a quantum particle in neighborhood of the event horizon, which is emitted in the Hawking radiation [24, 31–34]. On the other hand, gravity's rainbow holds the usual uncertainty principle [35, 36]. It is possible to translate the uncertainty principle $\Delta p \geq 1/\Delta x$ into a bound on the energy $E \geq 1/\Delta x$ which E can be interpreted as the energy of a particle emitted in the Hawking radiation. It has been shown that the uncertainty in the position of a test particle in the vicinity of the horizon should be equal to the event horizon radius [24, 31–34]

$$E \geq 1/\Delta x \approx 1/r_+, \quad (5)$$

where E is the energy of a particle near the horizon which bounded by the Planck energy E_P and can not increase to arbitrary values. This bound on the energy modifies temperature and entropy of the black hole in gravity's rainbow [24].

Now, we present various motivations for considering nonlinear electrodynamics. As we know, most physical systems are inherently nonlinear in the nature and nonlinear field theories are appropriate tools to investigate such systems. The main reason to consider the nonlinear electrodynamics (NED) comes from the fact that these theories are considerably richer than the Maxwell theory and in special case they reduce to the linear Maxwell field. In addition, some limitations of the Maxwell field such as, radiation inside specific materials [37–40] and description of the self-interaction of virtual electron-positron pairs [41–43], motivate one to regard NED [44, 45]. Also, taking into account the NED, one can remove both the big bang and black hole singularities [46–51]. One can find regular black hole solutions of Einstein gravity in the presence of a suitable NED [46, 47, 52–55]. Moreover, the effects of NED become indeed very important in superstrongly magnetized compact objects, such as pulsars and neutron stars [56–58]. In addition, horizonless magnetic solutions in presence of different nonlinear electromagnetic fields have been investigated in literature [59, 60]. Besides, an interesting property which is common to all NED models is the fact that black object solutions coupled to the NED models enjoy the zeroth and first laws of thermodynamics.

It is well-known that the electric field of a point-like charge has a divergency in the origin. To remove this singularity, about eighty years ago Born and Infeld introduced an interesting kind of NED which is known as Born-Infeld (BI) nonlinear electrodynamics (BINED) theory [61, 62]. Then, Hoffmann tried to couple the NED with gravity [63]. The gravitational fields coupled to BINED have been investigated for static black holes [64–69], rotating black objects [70–74], wormholes [75–78], and superconductors [79–84]. Also, BINED has acquired a new impetus, since it naturally arises in the low-energy limit of the open string theory [85–90]. Recently, two different BI type models of the NED with logarithmic [52] and exponential forms [91] have been introduced, which can also remove the divergency of the electric field near the origin. The logarithmic NED (LNED), like BI theory, removes divergency of the electric field, while the exponential NED (ENED) does not cancel the divergency, but its singularity is much weaker than that in the Maxwell theory. Black object solutions coupled to LNED and ENED have been studied in literature (for e.g., see [78, 92, 93]). Despite of BI type models, another example of NED is power Maxwell invariant (PMI) field [94–100]. The basic motivation of regarding PMI theory comes from the fact that it is interesting to modify Maxwell theory in such a way that its corresponding energy-momentum tensor will be conformally invariant. Taking into account the traceless energy-momentum tensor of electrodynamics, one should regard conformally invariant Maxwell field which is a subclass of PMI theory.

In this paper, we will study thermal stability of black holes in gravity's rainbow and see how the presence of rainbow functions modifies stability conditions and phase transition of black holes. Thermodynamical aspects of black holes have been among the most interesting subjects since the pioneering work of Hawking and Beckenstein [101–103]. The analogy between geometrical properties of the black holes and thermodynamical variables presents a deep insight into

relations between physical properties of gravity and classical thermodynamics. Due to this fundamental relation, it is believed that a consistent theory of quantum gravity could be derived through the use of thermodynamics of black holes. One of the important subjects of black hole thermodynamics is thermal stability. In thermal stability the positivity of heat capacity determines that the black holes are thermally stable. In addition, the divergency of the heat capacity is a place in which black hole meets a second order phase transition [104, 105]. Recently, it was shown that divergencies of the heat capacity also coincide with phase transitions that are observed in extended phase space [106].

The outline of the paper is as follow. Next section is devoted to introducing field equations and their related metric functions. In Sec. III, conserved and thermodynamic quantities will be obtained and the validity of the first law of thermodynamics will be examined. Then, the stability of the solutions and phase transition are investigated through the canonical ensemble. This paper will be finished with some final remarks.

II. FIELD EQUATIONS AND METRIC FUNCTION

The metric describing gravity's rainbow is constructed by considering the effects of energy of a particle. In other words, using doubly general relativity and parameterizing spacetime with the ratio of $\varepsilon = E/E_p$, one can construct a rainbow spacetime. Interestingly, this metric contains specific restriction with regard to mentioned ratio which will be stated later. Considering mentioned method for building up metric, one can have the rainbow metric in following form in four dimensions

$$d\tau^2 = -ds^2 = \frac{\Psi(r)}{f(\varepsilon)^2} dt^2 - \frac{1}{g(\varepsilon)^2} \left(\frac{dr^2}{\Psi(r)} + r^2 d\Omega_k^2 \right), \quad (6)$$

where $d\Omega_k^2$ represents the line elements of 2-dimensional hypersurfaces with the constant curvature $2k$ and volume V_2 with following forms

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2 \theta d\varphi^2 & k = 1 \\ d\theta^2 + \sinh^2 \theta d\varphi^2 & k = -1 \\ d\theta^2 + d\varphi^2 & k = 0 \end{cases}, \quad (7)$$

in which 2-dimensional hypersurface with plane, spherical and hyperbola symmetries are, respectively, denoted by $k = 0$, $k = 1$ and $k = -1$.

Our goal is to obtain rainbow solutions in Einstein gravity with cosmological constant in presence of NED. So the total Lagrangian for this system is

$$L_{\text{total}} = L_E - 2\Lambda + L(\mathcal{F}), \quad (8)$$

in which the Lagrangian of Einstein gravity is $L_E = R$, and Λ refers to the cosmological constant. The last term in Eq. (8) is the Lagrangian of NED, which we consider to be in following forms

$$L(\mathcal{F}) = \begin{cases} 4\beta^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2\beta^2}} \right), & \text{BINED} \\ \beta^2 \left[\exp\left(-\frac{\mathcal{F}}{\beta^2}\right) - 1 \right], & \text{ENED} \\ -8\beta^2 \ln \left(1 + \frac{\mathcal{F}}{8\beta^2} \right), & \text{LNED} \\ (-\mathcal{F})^s, & \text{PMI} \end{cases}, \quad (9)$$

where β and s are nonlinearity parameters, the Maxwell invariant is $\mathcal{F} = F_{ab}F^{ab}$ in which $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field tensor and A_b is the gauge potential. It is worthwhile to mention that in essence BINED, ENED and LNED are categorized under a same branch and they are called BI type models of NED. The series expansion of BI type models for large values of nonlinearity parameter yields similar results: first term is Maxwell invariant which is related to Maxwell theory of electromagnetic field, the second term is quadratic Maxwell invariant coupled with nonlinearity parameter and some factors which depend on theory under consideration (for explicit forms of the

expansion see Ref. [91]). On the other hand, PMI has different structure and properties comparing to BI type models. In order to recover the Maxwell field, one should set $s = 1$.

Now, we are in a position to obtain field equations. Applying variational principle to the Lagrangian (8), one can find

$$\nabla_a (\sqrt{-g} L_{\mathcal{F}} F^{ab}) = 0, \quad (10)$$

$$\Lambda g_{ab} + G_{ab}^{(E)} = \frac{1}{2} g_{ab} L(\mathcal{F}) - 2 L_{\mathcal{F}} F_{ac} F_b^c, \quad (11)$$

where $L_{\mathcal{F}} = \frac{dL(\mathcal{F})}{d\mathcal{F}}$ and $G_{ab}^{(E)} = R_{ab} - \frac{1}{2} g_{ab} R$.

Next, due to our interest in electrically charged black holes in gravity's rainbow, we consider a radial electric field which its related gauge potential is

$$A_b = h(r) \delta_b^t, \quad (12)$$

Using Eqs. (6), (9) and (10), we obtain following differential equations

$$\begin{aligned} r\beta^2 H' - 2f(\varepsilon)^2 g(\varepsilon)^2 H^3 + 2\beta^2 H &= 0, & \text{BINED} \\ r\beta^2 H' + 4rf(\varepsilon)^2 g(\varepsilon)^2 H^2 + 2\beta^2 H &= 0, & \text{ENED} \\ (4r\beta^2 + rf(\varepsilon)^2 g(\varepsilon)^2 H^2) H' + 8H\beta^2 - 2f(\varepsilon)^2 g(\varepsilon)^2 H^3 &= 0, & \text{LNED} \\ 2H + (2rs - 1) H' &= 0, & \text{PMI} \end{aligned} \quad (13)$$

in which $H = H(r) = \frac{dh(r)}{dr}$, and prime denotes derivation with respect to radial coordinate. It is a matter of calculation to show that

$$H(r) = \begin{cases} \frac{q}{r^2 \Gamma}, & \text{BINED} \\ \frac{q}{r^2} \exp\left(-\frac{1}{2} L_w\right), & \text{ENED} \\ \frac{2\beta^2 r^2}{qf(\varepsilon)^2 g(\varepsilon)^2} (\Gamma - 1), & \text{LNED} \\ \frac{q}{r^{\frac{2}{2s-1}}}, & \text{PMI} \end{cases}, \quad (14)$$

where $L_w = \text{Lamber}W\left(\frac{4q^2 f(\varepsilon)^2 g(\varepsilon)^2}{\beta^2 r^4}\right)$, $\Gamma = \sqrt{1 + \frac{q^2 f(\varepsilon)^2 g(\varepsilon)^2}{\beta^2 r^4}}$ and q is an integration constant related to the electric charge. In order to have a well-defined solution with PMI source, we should consider the PMI parameter, s larger than $1/2$ ($s > 1/2$).

By employing Eqs. (6), (11) and (14), one can find metric function for gravity's rainbow in presence of the mentioned NED as

$$\Psi(r) = k - \frac{m}{r} - \frac{\Lambda r^2}{3g(\varepsilon)^2} + \Upsilon, \quad (15)$$

with

$$\Upsilon = \begin{cases} \frac{2\beta^2 r^2}{3g(\varepsilon)^2} [1 - \mathfrak{F}_1], & \text{BINED} \\ \frac{\beta^2 r^2}{6g(\varepsilon)^2} \left[\frac{8\sqrt{\Gamma^2 - 1}}{5} \left(L_w^{\frac{3}{2}} \mathfrak{F}_3 + \frac{5(1+L_w)}{4\sqrt{L_w}} \right) - 1 \right], & \text{ENED} \\ \frac{4\beta^2 r^2 (\Gamma - 1)}{3g(\varepsilon)^2} \left[5 - \frac{\ln\left(\frac{2}{\Gamma + 1}\right)}{(\Gamma - 1)} + 4(\Gamma - 1) \mathfrak{F}_2 \right], & \text{LNED} \\ -\frac{r^2 (2s-1)^2}{(4s-6)g(\varepsilon)^2} \left(-\frac{\sqrt{2}(2s-3)qf(\varepsilon)g(\varepsilon)}{(2s-1)r^{2/(2s-1)}} \right)^{2s}, & \text{PMI} \end{cases},$$

where $\mathfrak{F}_1 = {}_2F_1\left(\left[\frac{-1}{2}, \frac{-3}{4}\right], \left[\frac{1}{4}\right], 1 - \Gamma^2\right)$, $\mathfrak{F}_2 = {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right], 1 - \Gamma^2\right)$ and $\mathfrak{F}_3 = {}_2F_1\left([1], \left[\frac{9}{4}\right], \frac{L_w}{4}\right)$ are the hypergeometric functions, and also m is an integration constant related to total mass of the solutions.

III. CONSERVED AND THERMODYNAMIC QUANTITIES

Considering obtained solutions for different models of NED, this section is devoted to calculating conserved and thermodynamical quantities, and study the effects of gravity's rainbow. Then, we are going to expand our study to stability of the solutions in canonical ensemble.

Due to the fact that employed metric only contains one temporal killing vector, one can use the concept of surface gravity for calculating the temperature on the event horizon (r_+) which leads to

$$T = \frac{1}{2\pi} \sqrt{\nabla_\mu \chi_\nu \nabla^\mu \chi^\nu} = \frac{1}{4\pi} \frac{g(\varepsilon)}{f(\varepsilon)} \frac{d\Psi(r)}{dr} \Big|_{r=r_+}, \quad (16)$$

where dependency on the rainbow functions indicates that the temperature is modified. Considering Eqs. (15) and (16), one can find

$$T = \begin{cases} \frac{kr_+^2 g(\varepsilon)^2 - 2r_+^4 \Lambda + 2\beta^2 r_+^4 (1 - \mathfrak{F}_{1+}) - 4q^2 f(\varepsilon)^2 g(\varepsilon)^2 \mathfrak{F}_{2+}}{4\pi f(\varepsilon) g(\varepsilon) r_+^3}, & \text{BINED} \\ \frac{\left[kg(\varepsilon)^2 - r_+^2 \left(\Lambda + \frac{\beta^2}{2} \right) \right] + \frac{4\beta q f(\varepsilon) g(\varepsilon) L_{w+}^{3/2}}{15(1+L_{w+})} \left[(L_{w+} - 5) \mathfrak{F}_{3+} - \frac{4}{9} L_{w+} \mathfrak{F}_{4+} + \frac{5}{4} \left(1 + \frac{3}{L_{w+}^2} \right) \right]}{4\pi f(\varepsilon) g(\varepsilon) r_+}, & \text{ENED} \\ \frac{\left[\beta^2 \ln \left(\frac{1+\Gamma_+}{2} \right) - \Lambda + \frac{kg(\varepsilon)^2}{r_+^2} - \frac{(\Gamma_+^2 - 1)^2}{r_+} \left(\frac{8\mathfrak{F}_5}{45\Gamma_+} + \frac{2\beta^2}{9(\Gamma_+^2 - 1)} \left[\frac{(7+5\Gamma_+)}{(1-\Gamma_+)\Gamma_+} + 2\mathfrak{F}_{2+} \right] \right) + \frac{(\beta^2 r_+^4 (5\Gamma_+^2 - \Gamma_+^6) + 1)}{3(1-\Gamma_+)r_+^5} \right]}{\pi f(\varepsilon) g(\varepsilon)} r_+^2, & \text{LNED} \\ \frac{\left[\frac{kg(\varepsilon)^2}{r_+^2} - \Lambda - \frac{(2s-1)}{2} \left(\frac{\sqrt{2}q f(\varepsilon) g(\varepsilon)(2s-3)}{(2s-1)r_+^{2/(2s-1)}} \right)^{2s} \right]}{4\pi f(\varepsilon) g(\varepsilon)} r_+, & \text{PMI} \end{cases}, \quad (17)$$

where $\Gamma_+ = \Gamma|_{r=r_+}$, $L_{w+} = L_w|_{r=r_+}$, $\mathfrak{F}_{1+} = \mathfrak{F}_1|_{r=r_+}$, $\mathfrak{F}_{2+} = \mathfrak{F}_2|_{r=r_+}$, $\mathfrak{F}_{3+} = \mathfrak{F}_3|_{r=r_+}$, $\mathfrak{F}_{4+} = {}_2F_1\left([2], \left[\frac{13}{4}\right], \frac{L_{w+}}{4}\right)$ and $\mathfrak{F}_5 = {}_2F_1\left(\left[\frac{3}{2}, \frac{5}{4}\right], \left[\frac{9}{4}\right], 1 - \Gamma_+^2\right)$.

In order to study entropy of the solutions, one can use the area law. It is easy to show that the entropy is

$$S = \frac{r_+^2}{4g(\varepsilon)^2}. \quad (18)$$

On the other hand, even with the modifications in metric, the entropy is independent of the electromagnetic fields. It is notable that although there is no trace of electromagnetic field in explicit form of the entropy, horizon radius is affected by electromagnetic field under consideration.

As for total charge of the solutions, one can employ the Gauss law. Considering this approach, one can show that for BI type models, results are the same. In other words, in case of BI type models, the total electric charge is independent of nonlinearity parameter. Whereas for the PMI case, the total charge is modified and depends on PMI parameter

$$Q = \begin{cases} \frac{f(\varepsilon)}{4\pi g(\varepsilon)} q, & \text{BI type models} \\ \frac{s(2s-1) \left(\frac{\sqrt{2}(2s-3)f(\varepsilon)g(\varepsilon)}{(2s-1)} \right)^{2s} q^{2s-1}}{8(3-2s)\pi f(\varepsilon)g(\varepsilon)^3}, & \text{PMI} \end{cases}. \quad (19)$$

In order to obtain electric potential, we can calculate it on the horizon with respect to a reference

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}, \quad (20)$$

which leads to

$$U = \begin{cases} \frac{q}{r_+} \mathfrak{F}_{2+}, & \text{BINED} \\ \frac{4\beta\sqrt{L_w} + \left[\frac{45(3+L_w)}{8} + \frac{9L_w + (4+L_w)\mathfrak{F}_{3+}}{4} + L_w^2 \mathfrak{F}_4 \right] r_+}{135g(\varepsilon)f(\varepsilon)(1+L_w)}, & \text{ENED} \\ \frac{8\beta(\Gamma_+ - 1) \left[10\mathfrak{F}_{2+} + (\Gamma_+^2 - 1)\mathfrak{F}_5 \right] r_+}{45q} + \frac{2\beta[6(1-\Gamma_+) + (8-5\Gamma_+)]r_+}{9qf\Gamma_+}, & \text{LNED} \\ qr_+^{\frac{2s-3}{2s-1}}, & \text{PMI} \end{cases}. \quad (21)$$

It is worthwhile to mention that for $s \geq \frac{3}{2}$, the gauge potential is not well-behaved, asymptotically. Therefore, we have both upper and lower limits for the nonlinearity parameter of PMI theory ($\frac{1}{2} < s < \frac{3}{2}$).

It is straightforward to show that the total finite mass of this black hole is

$$M = \frac{m}{8\pi f(\varepsilon)g(\varepsilon)}. \quad (22)$$

Here, we give more details for examination of the first law of thermodynamics. Evaluating the metric function on the event horizon ($\psi(r = r_+) = 0$), one can obtain geometrical mass (m) as a function of r_+ and q . Inserting $m(r_+, q)$ in Eq. (22), one finds

$$M = \begin{cases} \frac{\left[\frac{3k}{2}g(\varepsilon)^2 + r_+^2 \left(\beta^2 \left[1 - \mathfrak{F}_{1+} \right] - \frac{\Lambda}{2} \right) \right] r_+}{12\pi f(\varepsilon)g(\varepsilon)^3}, & \text{BINED} \\ \frac{\left[kg(\varepsilon)^2 - r_+^2 \left(\Lambda + \frac{\beta^2}{2} \right) + \frac{\beta q f(\varepsilon)g(\varepsilon)}{3\sqrt{L_w}} \left((1+L_w) + \frac{4}{5}L_w^2 \mathfrak{F}_{3+} \right) \right] r_+}{8\pi f(\varepsilon)g(\varepsilon)^3}, & \text{ENED} \\ \frac{\frac{9kr_+^2 g(\varepsilon)^2}{4} + \beta^2 r_+^4 \left[4(\Gamma_+^2 - 1)\mathfrak{F}_{2+} - 5\Gamma_+ + 3\ln(1+\Gamma_+) \right] - 3 \left[\frac{\Lambda}{4} + \beta^2 \left(\ln 2 - \frac{5}{3} \right) \right]}{18\pi r_+ f(\varepsilon)g(\varepsilon)^3}, & \text{LNED} \\ \frac{\left[2 \left(kg(\varepsilon)^2 - \frac{\Lambda r_+^2}{3} \right) - \frac{(2s-1)^2}{(2s-3)} \left(\frac{-\sqrt{2}(2s-3)qf(\varepsilon)g(\varepsilon)}{(2s-1)r_+^{2/(2s-1)}} \right)^{2s} \right] r_+}{16\pi f(\varepsilon)g(\varepsilon)^3}, & \text{PMI} \end{cases}. \quad (23)$$

Now, we use Eqs. (18) and (19) to obtain $M = M(S, Q)$ with the following forms

$$M = \begin{cases} \frac{2 \left[\frac{3k}{8} + S \left(\beta^2 - \frac{\Lambda}{2} \right) - S\beta^2 \mathfrak{F}_{1\Delta} \right] \sqrt{S}}{3\pi f(\varepsilon)}, & \text{BINED} \\ \frac{4 \left[\pi\beta Q \mathcal{L}_w^{3/2} \mathfrak{F}_{3\mathcal{L}} + \left(\frac{15k}{16} - \frac{5S}{8} (2\Lambda + \beta^2) \right) + \frac{5\pi\beta Q(1+\mathcal{L}_w)}{4\sqrt{\mathcal{L}_w}} \right] \sqrt{S}}{15\pi f(\varepsilon)}, & \text{ENED} \\ \frac{-16\pi^2 Q^2 \mathfrak{F}_{2\Delta} + \frac{3S(4S\Lambda - 3k)}{4} + 12S^2 \beta^2 \left[\ln \left(\frac{2}{1+\Delta} \right) + \frac{5(\Delta-1)}{3} \right]}{9\pi f(\varepsilon)\sqrt{S}}, & \text{LNED} \\ \frac{\left[\frac{3k}{4} - S\Lambda - \frac{3S(-2)^s(2s-1)^2}{2(2s-3)} \left(\frac{-\pi Q f(\varepsilon)^{s(3-2s)}}{s2^{s-2}S} \right)^{2s/(2s-1)} \right] \sqrt{S}}{3\pi f(\varepsilon)}, & \text{PMI} \end{cases}, \quad (24)$$

where $\mathfrak{F}_{1\Delta} = \mathfrak{F}_1|_{\Gamma=\Delta}$, $\mathfrak{F}_{2\Delta} = \mathfrak{F}_2|_{\Gamma=\Delta}$ and $\mathfrak{F}_{3\mathcal{L}} = \mathfrak{F}_3|_{L_w=\mathcal{L}_w}$. Also Δ and L_w are in the following forms

$$\Delta = \sqrt{1 + \frac{\pi^2 Q^2}{\beta^2 S^2}},$$

$$\mathcal{L}_w = \text{Lamber}W \left(\frac{4\pi^2 Q^2}{\beta^2 S^2} \right).$$

Now, we are in a position to study the validity of the first law of thermodynamics. Here, we should calculate $\left(\frac{\partial M}{\partial S} \right)_Q$ and $\left(\frac{\partial M}{\partial Q} \right)_S$, and then use Eqs. (18) and (19) to convert them as functions of r_+ and q . After some simplifications,

one finds that these quantities are, respectively, the same as temperature and potential which were obtained in Eqs. (17) and (21). Hence, although rainbow functions affected thermodynamic and conserved quantities, the first law remains valid as

$$dM = \left(\frac{\partial M}{\partial S} \right)_Q dS + \left(\frac{\partial M}{\partial Q} \right)_S dQ. \quad (25)$$

IV. THERMODYNAMIC STABILITY

In this section, we investigate thermal stability conditions of nonlinearly charged black hole solutions in gravity's rainbow. To do so, we investigate the heat capacity which is known as canonical ensemble. Thermal stability conditions are indicated by the sign of heat capacity. The positivity of heat capacity guarantees thermally stable solutions, whereas the negative heat capacity is denoted as an unstable state. Another advantage of studying the heat capacity is investigation of the phase transition point. There is a limitation for horizon radius (r_{+0}) as well as a phase transition that are obtainable through calculating the root and divergence point of the heat capacity.

One can use following relation for calculating the heat capacity

$$C_Q = T \left(\frac{\partial S}{\partial T} \right)_Q = T \frac{\left(\frac{\partial S}{\partial r_+} \right)_Q}{\left(\frac{\partial T}{\partial r_+} \right)_Q}. \quad (26)$$

Using obtained values for temperature and entropy for various models of NED (Eqs. (17) and (18)), one can find following relations

$$C_Q = \begin{cases} \frac{\left[\mathfrak{F}_{1+} + 2(\Gamma_+^2 - 1)\mathfrak{F}_{2+} - \frac{(kg(\varepsilon)^2 + (2\beta^2 - \Lambda)r_+^2)}{2\beta^2 r_+^2} \right]}{4\pi\beta^2 g(\varepsilon)^2 \left[\mathfrak{F}_{1+} + \frac{4(\Gamma_+^2 - 1)^2}{5}\mathfrak{F}_{5+} + \frac{(kg(\varepsilon)^2 - (2\beta^2 - \Lambda)r_+^2)}{2\beta^2 r_+^2} \right] r_+^2}, & \text{BINED} \\ \frac{r_+^2 (1 + L_{w+})^2 \mathcal{A}_1}{4\pi g^2(\varepsilon) \mathcal{A}_2}, & \text{ENED} \\ \frac{\beta^2 \Gamma_+^5 (1 - \Gamma_+) (\Gamma_+^2 - 1) \mathcal{B}_1 r_+^2}{128\pi g^2(\varepsilon) (\mathcal{B}_2 + \mathcal{B}_3)}, & \text{LNED} \\ \frac{\left(\left(s - \frac{1}{2} \right) \left[\frac{-\sqrt{2} q f(\varepsilon) g(\varepsilon) (2s-3)}{(2s-1)r_+^{2/(2s-1)}} \right]^{2s} - \Lambda + \frac{kg(\varepsilon)^2}{r_+^2} \right) r_+^2}{4\pi g(\varepsilon)^2 \left(\left(s + \frac{1}{2} \right) \left[\frac{-\sqrt{2} q f(\varepsilon) g(\varepsilon) (2s-3)}{(2s-1)r_+^{2/(2s-1)}} \right]^{2s} - \Lambda - \frac{kg(\varepsilon)^2}{r_+^2} \right)}, & \text{PMI} \end{cases}, \quad (27)$$

where A_1 , A_2 , B_1 , B_2 and B_3 are

$$\begin{aligned} \mathcal{A}_1 &= \beta q f(\varepsilon) g(\varepsilon) L_{w+}^2 \left[(5 - L_{w+}) \mathfrak{F}_{3+} + \frac{4L_{w+} \mathfrak{F}_4}{9} - \frac{5(3 + L_{w+}^2)}{4L_{w+}^2} \right] \\ &\quad - \frac{15L_{w+}^{1/2} (1 + L_{w+}) [kg^2(\varepsilon) - r_+^2 (2\Lambda + \beta^2)]}{4}, \\ \mathcal{A}_2 &= -60\beta q f(\varepsilon) g(\varepsilon) L_{w+}^2 \left\{ (L_{w+}^3 + 3L_{w+}^2 - 9L_{w+} - 35) \mathfrak{F}_5 - \frac{16}{3} L_{w+} \left(L_{w+} + \frac{4}{3} \right) \mathfrak{F}_4 \right. \\ &\quad \left. - \frac{32}{117} L_{w+}^2 (L_{w+} + 1) \mathfrak{F}_6 + \frac{5(L_{w+} - 1)}{4L_{w+}^2} (L_{w+}^3 + 5L_{w+}^2 + 15L_{w+} + 3) \right\} \\ &\quad + \frac{15L_{w+}^{1/2} [kg^2(\varepsilon) + r_+^2 (2\Lambda + \beta^2)]}{8} (L_{w+}^3 + 3L_{w+}^2 + 3L_{w+} + 1), \\ \mathcal{B}_1 &= 9 \ln(2(1 + \Gamma_+)) + 4(1 - \Gamma_+^2) \left[\mathfrak{F}_{2+} + \frac{2}{5} (1 - \Gamma_+^2) \mathfrak{F}_5 \right] - \frac{9}{4\beta^2} \left(\Lambda - \frac{kg(\varepsilon)^2}{r_+^2} \right) \\ &\quad + \frac{3(6 - \Gamma_+ - 5\Gamma_+^3)}{(1 + \Gamma_+) \Gamma_+} + \frac{2(1 + \Gamma_+)(7 + 5\Gamma_+)}{\Gamma_+}, \end{aligned}$$

$$\begin{aligned}
\mathcal{B}_2 = & \frac{9\Gamma_+^3}{16\beta^6 r_+^4} \left[\frac{\beta^2 (3 - \Gamma_+^2)}{2} - (\Gamma_+^2 - \Gamma_+ - 1) \right] - \frac{\ln(1 + \Gamma_+) \Gamma_+^2}{3r_+^{20}} \\
& - \frac{\beta^6 \Gamma_+^5 (\Gamma_+^2 - 1)^2}{2} \left\{ \frac{\left(\frac{(1+5\Gamma_+^2)}{4} \mathfrak{F}_{2+} - \frac{(\Gamma_+^2 - 1)(3+5\Gamma_+^2)}{5} \mathfrak{F}_5 \right)}{(\Gamma_+^2 - 1)} + \frac{(\Gamma_+^2 - 1)(1 + \Gamma_+^2) \mathfrak{F}_7}{3\beta^2 g(\varepsilon)^4} \right. \\
& - \frac{9}{16\beta^4 \Gamma_+^2} \left[\frac{\left(\frac{17\beta^2}{3} - \frac{\Gamma_+^2}{r_+^4} - \frac{3(3-\Gamma_+^2)\ln 2}{4\beta^2 r_+^6} \right)}{(\Gamma_+^2 - 1)^2} + \frac{\left(\Lambda + \frac{kg(\varepsilon)^2}{r_+^2} + \frac{32\beta^2(\Gamma_+^2 - 1)}{3} \right)}{2(\Gamma_+^2 - 1)^2} \right. \\
& \left. \left. + \frac{3\left(\frac{kg(\varepsilon)^2}{r_+^2} - \frac{\Lambda}{3}\right)}{4} + \frac{\left(\Lambda + \frac{10kg(\varepsilon)^2}{3r_+^2} + \frac{320\beta^2(\Gamma_+^2 - 1)}{9} \right)}{4(\Gamma_+^2 - 1)} \right] \right\}, \\
\mathcal{B}_3 = & \frac{\beta^4 \Gamma_+^2}{3} \left\{ \frac{-45\Gamma_+^5}{32} - \frac{3(\Gamma_+^2 - 1)\Gamma_+}{10} \left[-\frac{5(\Gamma_+^2 - 1)\Gamma_+^2}{2} \left(\mathfrak{F}_{2+} + \frac{2(1 - \Gamma_+^2)}{5} \mathfrak{F}_5 \right) \right. \right. \\
& + \frac{45}{32\beta^2} \left(4 \left[3 - \beta^2 \Gamma_+^2 \ln 2 + \frac{kg(\varepsilon)^2}{4r_+^2} + \frac{23\beta^2(\Gamma_+^2 - 1)}{9} \right] + \frac{(\Gamma_+^2 - 1)[kg(\varepsilon)^2 - \Lambda r_+^2]}{r_+^2} \right) \left. \right] \\
& + (\Gamma_+^2 - 1)\Gamma_+^4 \left(\frac{9\mathfrak{F}_{2+}}{4} - \frac{12(\Gamma_+^2 - 1)\mathfrak{F}_5}{5} + (\Gamma_+^2 - 1)^2 \mathfrak{F}_7 \right) \\
& - \frac{27(\Gamma_+^2 - 1)^2}{64\beta^2} \left[\frac{4(2 + \beta^4 \Gamma_+^4 \ln 2)}{\beta^2(\Gamma_+^2 - 1)^2} + \frac{\left(\Lambda + \frac{kg(\varepsilon)^2}{r_+^2} + \frac{38\beta^2(\Gamma_+^2 - 1)}{3} \right)}{(\Gamma_+^2 - 1)^2} \right. \\
& \left. \left. + \frac{2\left(\Lambda + \frac{kg(\varepsilon)^2}{r_+^2} + \frac{34\beta^2(\Gamma_+^2 - 1)}{9} \right)}{(\Gamma_+^2 - 1)} + \frac{\left(\Lambda + \frac{kg(\varepsilon)^2}{r_+^2} + \frac{10\beta^2(\Gamma_+^2 - 1)}{9} \right)}{q^2} \right] \right\},
\end{aligned}$$

in which $F_7 = {}_2F_1\left(\left[\frac{5}{2}, \frac{9}{4}\right], \left[\frac{13}{4}\right], 1 - \Gamma_+^2\right)$.

In order to investigate thermal stability of the solutions, we may regard explicit functional forms of the rainbow functions $f(\varepsilon)$ and $g(\varepsilon)$. The choices of these functions are motivated from various theoretical and phenomenological considerations. Here, we refer to more important forms which are based on interesting phenomenology.

The first model is related to constant speed of light and one may use it to solve the horizon problem [15, 16]. The functional form of both rainbow functions are the same

$$f(\varepsilon) = g(\varepsilon) = \frac{1}{1 - \lambda\varepsilon}. \quad (28)$$

In addition, motivated by the results of loop quantum gravity and also non-commutative geometry, the rainbow functions are given by [107, 108]

$$f(\varepsilon) = 1, \quad g(\varepsilon) = \sqrt{1 - \eta\varepsilon^n}. \quad (29)$$

Also, it was shown that in non-commutative geometry context, it is better to regard a Gaussian trial functional form (exponential function) for $f(\varepsilon)$ to avoid a regularization/renormalization scheme [23, 109]. On the other hand, based on the hard spectra from gamma-ray burster's, one may consider the rainbow functions [4] with the following forms

$$f(\varepsilon) = \frac{e^{\xi\varepsilon} - 1}{\xi\varepsilon}, \quad g(\varepsilon) = 1. \quad (30)$$

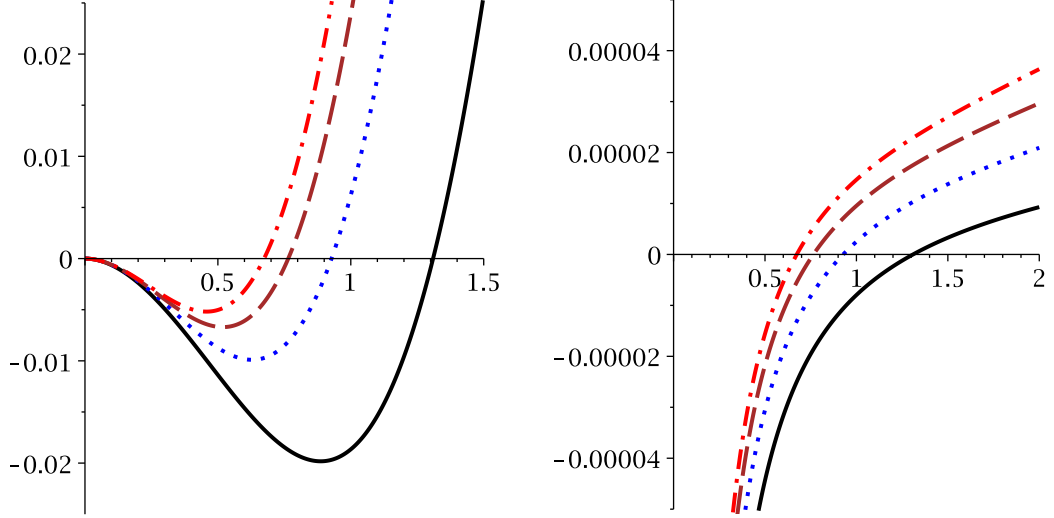


FIG. 1: **ENED branch:** C_Q (left panel) and T (right panel) versus r_+ for $k = 1$, $l = 1$, $\beta = 2$, $\varepsilon = 0.2$ and $q = 1$. $\xi = 0.5$ (continues line), $\xi = 1$ (dotted line), $\xi = 1.5$ (dashed line) and $\xi = 2$ (dotted-dashed line).

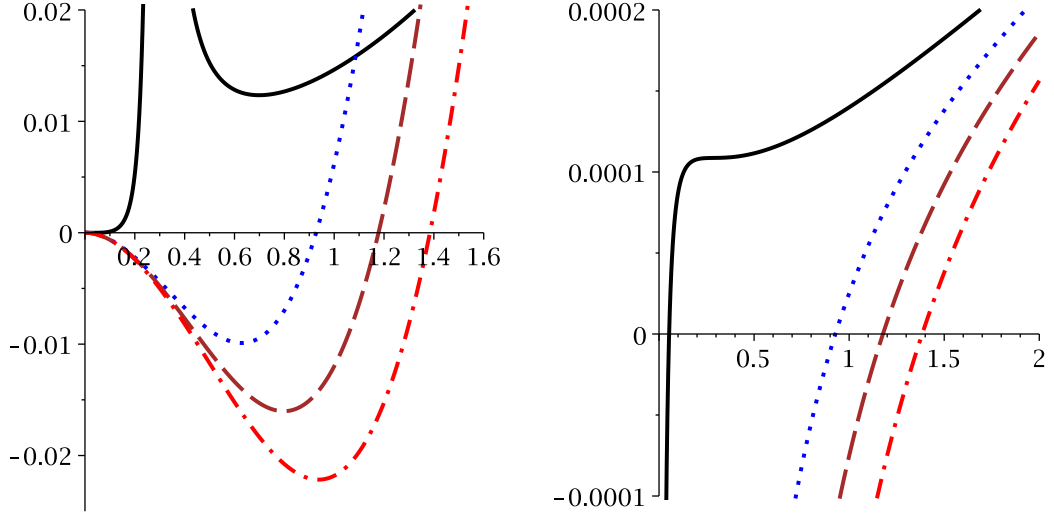


FIG. 2: **ENED branch:** C_Q (left panel) and T (right panel) versus r_+ for $k = 1$, $l = 1$, $\beta = 2$, $\varepsilon = 0.2$ and $\xi = 1$. $q = 0.1$ (continues line), $q = 1$ (dotted line), $q = 1.5$ (dashed line) and $q = 2$ (dotted-dashed line).

In order to study the thermodynamical behavior of the system, we use Eq. (30), in which $f(\varepsilon)$ has an exponential form. Considering these two rainbow functions, we plot following diagrams to study the effects of variation of different parameters on stability conditions and phase transition of the obtained solutions (Figs. 1-5).

In case of BI type models, for specific values of different parameters, there is a region in which the temperature and heat capacity are negative. Black hole solutions are not physical in this region. In this case, the heat capacity enjoys a root, r_{+0} , in which for $r_+ > r_{+0}$, BI type black holes are in stable state with positive temperature. r_{+0} is a decreasing function of ξ (Fig. 1) and an increasing function of electric charge (Fig. 2). Interestingly, for small values of q , heat capacity may enjoy a divergency which indicates a second order phase transition. Here, a phase transition of smaller to larger black holes takes place (left panel of Fig. 2). Surprisingly, the plotted diagram for temperature in this case, shows the existence of subcritical isobar. The presence of subcritical isobars is observed for Van der Waals like liquid/gass systems. Such behavior for black holes is only observed in cases of considering cosmological constant as thermodynamical pressure [106]. Here, without the use of analogy between cosmological constant and thermodynamical pressure, we found the properties of critical point.

In addition, we see that for suitable choices of different parameters and small values of nonlinearity parameter,

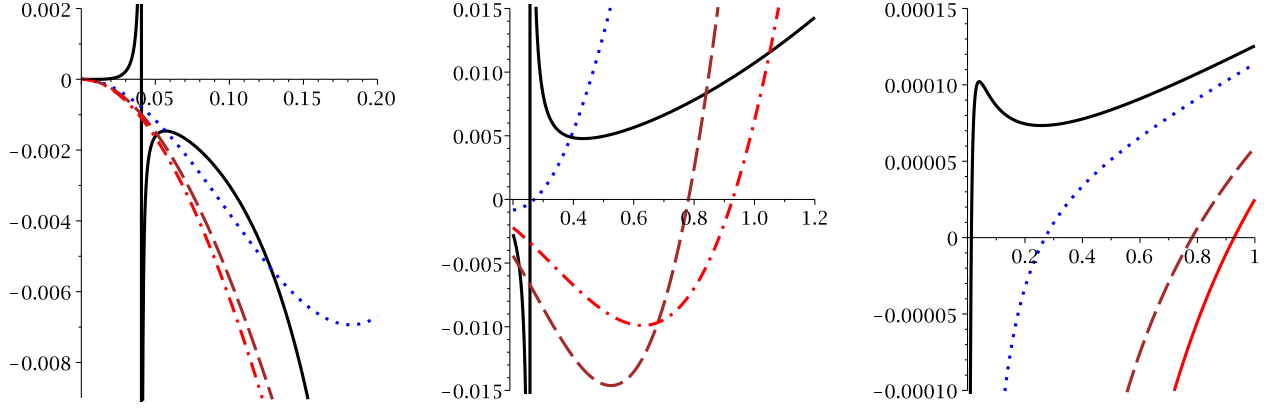


FIG. 3: **ENED branch:** C_Q (left and middle panels) and T (right panel) versus r_+ for $k = 1$, $l = 1$, $q = 1$, $\varepsilon = 0.2$ and $\xi = 1$ "for different scales".
 $\beta = 0.1$ (continues line), $\beta = 0.2$ (dotted line), $\beta = 1$ (dashed line) and $\beta = 2$ (dotted-dashed line).

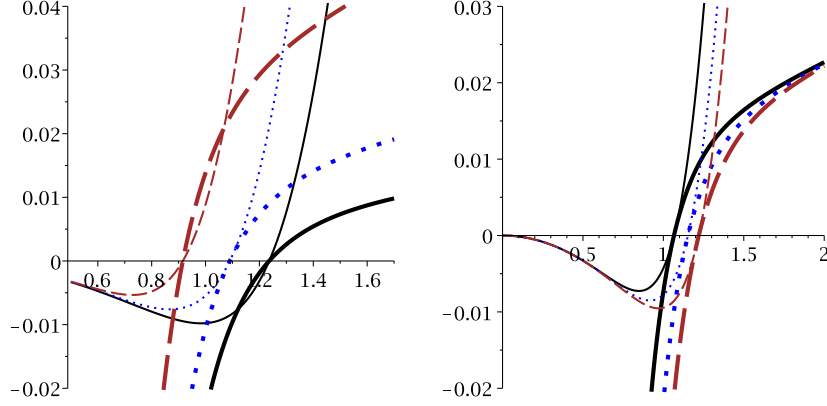


FIG. 4: **PMI branch:** C_Q and T (bold lines) versus r_+ for $k = 1$, $l = 1$, $\varepsilon = 0.2$ and $s = 0.7$.
 left panel: $q = 1$, $\xi = 0.5$ (continues line), $\xi = 1$ (dotted line) and $\xi = 5$ (dashed line).
 right panel: $\xi = 1$, $q = 0.9$ (continues line), $q = 1.3$ (dotted line) and $q = 1.7$ (dashed line).

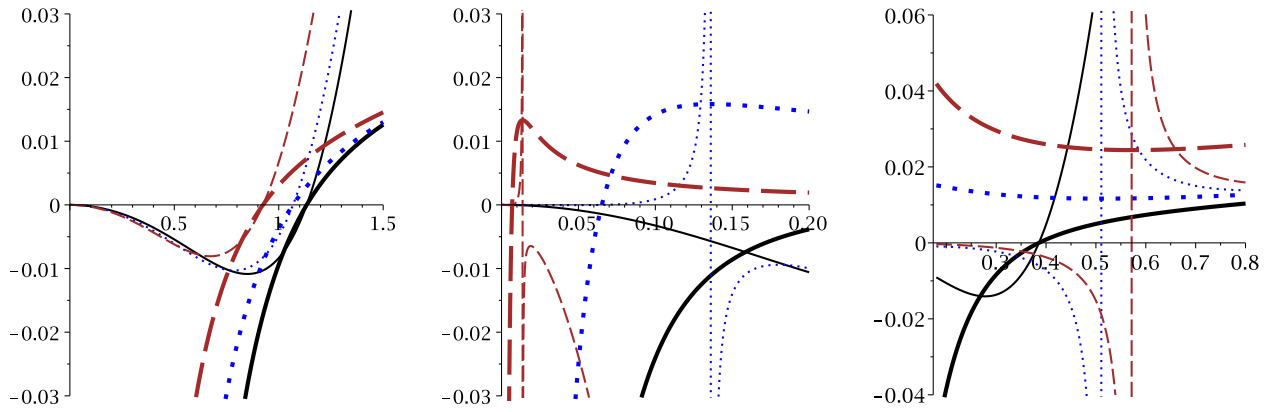


FIG. 5: **PMI branch:** C_Q and T (bold lines) versus r_+ for $k = 1$, $l = 1$, $\varepsilon = 0.2$, $q = 1$ and $\xi = 1$ "for different scales".
 left panel: $s = 0.9$ (continues line), $s = 1$ (dotted line) and $s = 1.1$ (dashed line).
 middle and right panels: $s = 1.3$ (continues line), $s = 1.4$ (dotted line) and $s = 1.45$ (dashed line).

there are one root and two extrema in temperature diagram; one minimum and one maximum. These extrema present themselves as divergencies in heat capacity diagrams. The stable states exist between the root and smaller divergency and after the larger divergence point (Fig. 3). Between two divergencies, the heat capacity is negative while the temperature is positive, and therefore, in this region unstable state exists. This instability switches to stable state as heat capacity meets the divergencies. In other words, a phase transition of smaller unstable to larger stable solutions takes place at the larger divergence point, while a phase transition of larger unstable to smaller stable occurs at the smaller divergence point. Increasing the nonlinearity parameter leads to vanishing of these phase transition points (Fig. 3).

It is worthwhile to mention that the small values of nonlinearity parameter represent strong nonlinearity for the system. Therefore, as the nonlinearity of the system increases (nonlinearity parameter decreases), the thermodynamical structure of the solutions will be modified. In addition, numerical evaluation shows that the root of heat capacity in LNED theory is bigger comparing to other theories of BI family. This indicates that physical stable solutions are obtained in higher values of horizon radius for this theory of NED comparing to BINED and ENED. In opposite, the divergence point in LNED is located at smaller horizon radius which shows that black hole solutions in presence of LNED acquire thermal stability faster comparing to other two BI models.

For PMI case, for specific values of different parameters, like BI case, a root is observed which is a limitation bound between non-physical and physical states. The value of root is a decreasing function of ξ (Fig. 4 left panel) and an increasing function of electric charge (Fig. 4 right panel). Interestingly, regarding the variation of s , for $0.5 < s < s_c$ ($s_c \approx 1.3$ for considered parameters), only a root (mentioned bound point) exists (Fig. 5 left panel). As for $s = s_c$, there are two extrema, one minimum and one maximum, and also one root for the temperature (Fig. 5 middle and right panels). Between root and smaller divergence point and after larger divergence point, heat capacity is positive and system is in stable state, whereas between two divergencies system has negative heat capacity (with positive T), hence, black holes are not stable. Another interesting effect is that for $s_c < s < 1.5$, the smaller divergence point is a decreasing function of s while larger divergence point is an increasing function of it (Fig. 5).

V. CLOSING REMARKS

In this paper, we have considered gravity's rainbow in presence of various models of NED. At first, 4-dimensional black hole solutions for these configurations were derived, and then, related conserved and thermodynamic quantities were calculated. It was shown that some of the conserved and thermodynamical quantities were modified due to the contribution of gravity's rainbow. Despite these modifications, the first law of thermodynamics was valid for these black hole solutions.

Next, we have studied the stability of the solutions and phase transition points in context of canonical ensemble. The employed nonlinear electromagnetic fields in this paper were categorized into two types: BI type includes Born-Infeld, logarithmic and exponential forms and PMI model which is power law generalization of Maxwell Lagrangian.

In case of BI types, we have found a lower bound for the horizon radius, r_{+0} , in which the black holes are not physical for $r_+ < r_{+0}$. Interestingly, for suitable choices of different parameters, we have found a second order phase transition point which had characteristics of $T - V$ diagrams for critical pressure (subcritical isobar was observed). Then, by employing different parameters, we have found two extrema and a root in temperature, and one root and two divergencies in heat capacity diagrams. In this case, there existed two phase transitions of smaller unstable to larger stable and larger unstable to smaller stable. The phase transitions took place at divergencies of the heat capacity. It was pointed out that the largest root and the smallest divergence point of the heat capacity belonged to the logarithmic form of NED.

For PMI case, similarly, stable physical and unstable non-physical states were observed for a range of s . Interestingly, for another range of this parameter the thermodynamical behavior was modified. A root, a maximum and a minimum were observed for the temperature. In places of these extrema (heat capacity enjoyed the existence of divergencies), two phase transitions of medium unstable to smaller or larger stable black holes took place. Another interesting property of this matter-field was the effects of variation of the s on divergencies of the heat capacity. For specific range of s , the smaller divergence point was a decreasing function of s while the larger divergence point was an increasing function of nonlinearity parameter.

It is evident that BI types and PMI models have different effects and contributions to thermodynamical behavior of the black hole system. In other words, considering these two classes of NED leads to different modifications and properties for the system. In case of BI family of NED, the theory under consideration will indicate the place of formation of the stable solutions.

It will be worthwhile to study obtained solutions in this paper in context of extended phase space and investigate both modifications of gravity's rainbow and nonlinear electromagnetic field on critical behavior of the system. In addition, generalization to higher dimensions is another interesting work. We left these issues for the forthcoming

work.

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